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Quantum mechanics of toroidal anyons

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Abstract. We consider a toroidal solenoid with an electric charge attached to it. It turns out that statistical properties of the wavefunction describing interacting toroidal anyons depend on both their relative position and orientation. The influence of the particular gauge choice on the exchange properties of the wavefunction is studied.

1. Introduction

Since Wilczek's profound papers [1] there has been increasing activity in the study of systems containing an electric charge and a magnetic flux. These composites, called anyons, carry fractional angular momentum and possess unusual statistics. Physically, they are manifested as quasiparticles in the fractional quantum Hall effect [2] and probably in high-temperature superconductivity [3]. Up to now only two-dimensional anyons have been considered (e.g. see [4]). In three-dimensional space an anyon can be realized as a cylindrical solenoid with an electric charge attached. It is the goal of the present paper to study the system composed of a toroidal solenoid and a charged particle. The plan of our exposition is as follows. In section 2 we write out the Lagrangian and the Schrödinger equation (SE) describing an interaction of two anyons without specifying the type of solenoid used. In section 3 the main facts concerning toroidal solenoids are presented. In section 4 the particular model of a toroidal anyon is proposed. The arguments for the existence of multivalued wavefunctions in the field of an impenetrable toroidal solenoid are given in section 5. Two interacting toroidal anyons are considered in section 6. It turns out that the statistical properties of their wavefunction depend on both the anyons' mutual separation and orientation. The influence of the particular gauge choice on the exchange properties of wavefunctions is studied in section 7. In the following section we explain why the arguments denying the existence of fractional statistics in three-dimensional space do not work in the treated case.

2. *Basic equations*

A composite consisting of a particle with a charge e and a solenoid with a magnetic flux ϕ is called an anyon [1]. The Lagrangian of this system is [5, 6]

$$L=\tfrac{1}{2}mv^2+\tfrac{1}{2}MV^2+\frac{e}{c}\,A(r-R)\cdot(v-V). \tag{2.1}$$

The notation m, $\boldsymbol{v}$ and M, $\boldsymbol{V}$ refers to the charged particle and the solenoid; $\boldsymbol{A}$ is the vector potential produced by the solenoid situated at $\boldsymbol{R}$ at the position $\boldsymbol{r}$ of the charged particle. The Lagrangian (2.1) describes both the Aharonov–Bohm and Aharonov–Casher [5] effects. The latter was recently [7] confirmed experimentally. Consider two anyons $(e_1\phi_1)$ and $(e_2\phi_2)$ with masses m_1 and m_2. They are described by the following Lagrangian [6]:

$$L=\tfrac{1}{2}m_1\boldsymbol{v}_1^2+\tfrac{1}{2}m_2\boldsymbol{v}_2^2+\frac{e}{c}\boldsymbol{A}\cdot\boldsymbol{v}-V. \tag{2.2}$$

Here we put $e\boldsymbol{A}\equiv e_1\boldsymbol{A}_{12}(\boldsymbol{r}_1-\boldsymbol{r}_2)-e_2\boldsymbol{A}_{21}(\boldsymbol{r}_2-\boldsymbol{r}_1)$; $\boldsymbol{A}_{12}(\boldsymbol{A}_{21})$ is the vector potential generated by the anyon 2(1) at the position of anyon 1(2); $\boldsymbol{v}=\boldsymbol{v}_1-\boldsymbol{v}_2$ and V is the electromagnetic interaction between charged particles. It is suggested here that the particular anyon feels only the electromagnetic field of the other. The self-interaction between the magnetic flux and the charge of the same anyon is disregarded, which is a routine operation in anyon theory. By quantizing the Lagrangian we obtain the following SE:

$$-\frac{\hbar^2}{2M}\nabla_R^2\Psi-\frac{\hbar^2}{2\mu}\left(\nabla_r-\frac{\mathrm{i}e}{\hbar c}\boldsymbol{A}\right)^2\Psi+V\Psi=E\Psi. \tag{2.3}$$

Here $M=m_1+m_2$, $\mu=m_1m_2/(m_1+m_2)$, $\boldsymbol{R}=(m_1\boldsymbol{r}_1+m_2\boldsymbol{r}_2)/(m_1+m_2)$, $\boldsymbol{r}=\boldsymbol{r}_1-\boldsymbol{r}_2$. If in addition $e_1=e_2=e$ and $m_1=m_2=m$, then this equation reduces to

$$\begin{gathered}-\frac{\hbar^2}{4m}\nabla_R^2\Psi-\frac{\hbar^2}{m}\left(\nabla_r-\frac{\mathrm{i}e}{\hbar c}\boldsymbol{A}\right)^2\Psi+V\Psi=E\Psi\\ \boldsymbol{A}=\boldsymbol{A}_{12}(\boldsymbol{r})-\boldsymbol{A}_{21}(-\boldsymbol{r}).\end{gathered} \tag{2.4}$$

Separating the centre-of-mass coordinates ($\Psi=\exp(\mathrm{i}\boldsymbol{KR})\Psi$) we get

$$\begin{gathered}\left(\nabla_r-\frac{\mathrm{i}e}{\hbar c}\boldsymbol{A}\right)^2\Psi+(\varepsilon-v)\Psi=0\\ \varepsilon=\frac{mE}{\hbar^2}-\frac{K^2}{4}\qquad v=\frac{mV}{\hbar^2}.\end{gathered} \tag{2.5}$$

Let the anyon's solenoid be toroidal. Similarly to the term 'cyon' used for the cylindrical anyon [8], the term 'toron' will be used for the toroidal anyon. Some facts concerning the toroidal solenoid which are needed for the subsequent discussion will be presented in the next section.

3. The electromagnetic field of the toroidal solenoid

The magnetic field of the toroidal solenoid $(\rho-d)^2+z^2=R^2$ equals $\boldsymbol{H}=\boldsymbol{e}_\varphi g/\rho$ inside the solenoid and zero outside it. The constant g is expressed through the magnetic flux ϕ: $g=\phi[2\pi(d-\sqrt{d^2-R^2})]^{-1}$. In the Coulomb gauge (div $\boldsymbol{A}=0$) the vector potential of the toroidal solenoid was obtained in [9]. Later it was used for the description of the electron scattering on the toroidal solenoid [10–13]. Here, we present its components only for the infinitely thin ($R\ll d$) solenoid:

$$\begin{gathered}A_z=\frac{\phi d}{2\pi}\frac{1}{(d\rho)^{3/2}}\frac{1}{\sinh\nu}[\rho Q^1_{1/2}(\cosh\nu)-dQ^1_{-1/2}(\cosh\nu)]\\ A_x=\frac{x}{\rho}A_\rho\qquad A_y=\frac{y}{\rho}A_\rho\\ A_\rho=-\frac{\phi d}{2\pi}\frac{1}{(d\rho)^{3/2}}\frac{1}{\sinh\nu}zQ^1_{1/2}(\cosh\nu).\end{gathered} \tag{3.1}$$

Here $\cosh \nu = (r^2+d^2)/2d\rho$ and Q^1_α are the Legendre functions of the second kind. At large distances $\boldsymbol{A}$ falls as r^{-3}:

$$A_x \approx \frac{3\pi g R^2 d}{4}\frac{xz}{r^5} \qquad A_y \approx \frac{3\pi g R^2 d}{4}\frac{yz}{r^5}$$

$$A_z \approx -\frac{\pi g R^2 d}{4}\frac{r^2-3z^2}{r^5}.$$

As outside the solenoid $\boldsymbol{H} = \operatorname{rot} \boldsymbol{A} = 0$, the vector potential may be presented as a gradient of some function χ: $\boldsymbol{A} = \operatorname{grad} \chi$. This function turns out to be multi-valued (more accurately, discontinuous) as $\oint \boldsymbol{A}\, \mathrm{d}\boldsymbol{x} = \phi$ for any contour passing through the solenoid's hole. To write this function explicitly we introduce the toroidal coordinates

$$x = a\frac{\sinh \mu \cos \varphi}{\cosh \mu - \cos \theta} \qquad y = a\frac{\sinh \mu \sin \varphi}{\cosh \mu - \cos \theta}$$

$$z = a\frac{\sin \theta}{\cosh \mu - \cos \theta} \tag{3.2}$$

$$(0<\mu<\infty, -\pi<\theta<\pi, 0<\varphi<2\pi).$$

Let $\mu = \mu_0$ correspond to the toroidal solenoid S. Then for $\mu > \mu_0 (<\mu_0)$ the point $P(x, y, z)$ (where x, y, z are given by (3.2)) lies inside (outside) S. For μ fixed (say, $\mu = \mu_0$) the points $P(x, y, z)$ fill the surface of the torus $(\rho-d)^2+z^2 = R^2$ with the parameters $d = a \coth \mu_0$, $R = a/\sinh \mu_0$. The value of the angle θ jumps from $-\pi$ to π when one intersects the circle of the radius $d-R$ lying in the $z=0$ plane. Now we are able to write out the χ function explicitly [9, 10]:

$$\chi = \chi_0(\theta) + \frac{4\sqrt{2}\, ga}{\pi}(\cosh \mu - \cos \theta)^{1/2} \sum_{n=1}^{\infty} \beta_n P_{n-1/2} \sin n\theta. \tag{3.3}$$

Here $P_{n-1/2}$ is the Legendre function of the first kind;

$$\beta_n = -\sum_{K=n}^{\infty} Q_{K-1/2}(0) Q_{K+1/2}(0)$$

χ_0 is given by

$$\chi_0 = \frac{1}{4\pi}\phi\left(-2\theta + \sum_{n=1}^{\infty}\frac{1}{n}\sin n\theta[P_{-1/2}(Q_{n+1/2}+Q_{n-3/2}) - 2P_{1/2}Q_{n-1/2}]\right).$$

From now we do not indicate the argument of the Legendre function if it equals $\cosh \mu$; further, $Q_\nu(0) \equiv Q_\nu(\cosh \mu_0)$, $P_\nu(0) \equiv P_\nu(\cosh \mu_0)$. Clearly, χ transforms into χ_0 for the infinitely thin ($R \ll d$ or $\mu_0 \gg 1$) solenoid. We see that χ suffers a jump from $-\frac{1}{2}\phi$ to $\frac{1}{2}\phi$ when one intersects the circle of the radius $d-R$ lying in the $z=0$ plane. At large distances χ falls as r^{-2} (this follows from (3.3)):

$$\chi \approx -\frac{\phi d R^2}{8(d-\sqrt{d^2-R^2})}\frac{\cos \theta_s}{r^2}$$

(r and θ_s are the usual spherical coordinates). The unitary transformation $\Psi = \Psi' \exp(\mathrm{i}e\chi/\hbar c)$ may be used to eliminate vector potentials outside the solenoid. The transformed wavefunction is multi-valued if the initial wavefunction Ψ is single-valued:

$$\Psi'(\rho \leqslant d-R, z=0-) = \Psi'(\rho \leqslant d-R, z=0+)\exp(-\mathrm{i}\gamma). \tag{3.4}$$

Here $\gamma = e\phi/\hbar c$. The reverse is also true.

For the arbitrary orientation of the solenoid the vector potential at the point $\boldsymbol{r}$ is given by

$$A_i(\boldsymbol{r})=\sum R_{iK}(\varphi,\theta,\psi)A_K(\boldsymbol{r}).$$

Here A_K are given by (3.1), R is the usual rotation matrix and φ, θ, ψ are the angles defining the orientation of the solenoid fixed frame with respect to the laboratory one.

4. The particular realization of a toron

Usually, in treating the anyon problem one does not specify the way in which the charge is attached to the solenoid. One of the possible ways to do this is to charge the surface ($\mu=\mu_0$) of the toroidal solenoid. To exclude the appearance of the currents at the solenoid surface the latter should be at the constant electrostatic potential ϕ_0. The elementary calculations show [14] that the electric charge should be distributed over the solenoid surface with the density

$$\rho(\theta)=\delta(\mu-\mu_0)\frac{(\cosh\mu_0-\cos\theta)^{5/2}}{\sqrt{2}\,\pi^2a^2}\frac{\phi_0}{\sinh\mu_0}\sum_{n=0}^{\infty}\frac{\cos n\theta}{1+\delta_{n0}}(P_{n-1/2}(0))^{-1}.$$

From now we suggest that the summation, if it is not specified, extends from $n=0$ to $n=\infty$. The electrostatic potential generated by this density equals ϕ_0 inside the toroidal solenoid ($\mu>\mu_0$) and

$$\phi=\frac{2\sqrt{2}}{\pi}\phi_0(\cosh\mu-\cos\theta)^{1/2}\sum P_{n-1/2}\frac{Q_{n-1/2}(0)}{P_{n-1/2}(0)}\frac{\cos n\theta}{1+\delta_{n0}}$$

outside it. The total surface charge e may be expressed through ϕ_0:

$$e=\frac{4a\phi_0}{\pi}\sum\frac{1}{1+\delta_{n0}}\frac{Q_{n-1/2}(0)}{P_{n-1/2}(0)}.$$

The subsequent consideration does not depend on this particular realization.

5. The possibility of multi-valued wavefunctions in the field of a toroidal solenoid

We present here the arguments for the existence of multi-valued wavefunctions in the magnetic field of an impenetrable toroidal solenoid. But, firstly, we repeat similar arguments [15–17] for the well-known case of a cylindrical solenoid. Consider two identical charged particles 1 and 2 in the field of an infinite cylindrical solenoid (figure 1). Now we exchange particles 1 and 2. This procedure is path dependent if multi-valued wavefunctions are used. The wavefunctions remain the same if there is no net magnetic flux inside the closed contour composed of exchange paths 1 and 2: $\Psi(2,1)=\Psi(1,2)$. On the other hand, the wavefunctions change when the finite magnetic flux ϕ is presented inside the above closed contour $\Psi(2,1)=\Psi(1,2)\exp(\mathrm{i}\gamma)$, $\gamma=e\phi/\hbar c$. If $\gamma=2\pi n$ then $\Psi(2,1)=\Psi(1,2)$, i.e. the presence of the magnetic flux does not affect the exchange properties of the wavefunctions. When $\gamma=\pi(2n+1)$ one has $\Psi(2,1)=-\Psi(1,2)$, that is, the particles behave as fermions. For arbitrary γ one has intermediate statistics (between bosons and fermions). The impenetrability of the solenoid guarantees that exchange paths shown at the lower part of figure 1 cannot be continuously deformed (or shrunk to a point) into that presented at the upper part of the same figure.

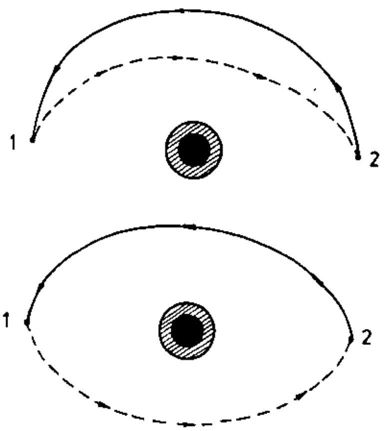

Figure 1. The trivial (upper part) and non-trivial (lower part) exchange paths in the field of a cylindrical solenoid. The space region where $H \neq 0$ is blacked in. The inaccessible region is hatched.

Now we turn to the behaviour of wavefunctions in the magnetic field of an impenetrable toroidal solenoid. In figure 2 there are shown exchange paths which do not embrace the magnetic flux ϕ. Each of them can be contracted to a point without intersecting the impenetrable torus. Thus, $\psi(2,1) = \psi(1,2)$ for them. Some of the topologically non-trivial exchange paths are shown in figure 3. None of them can be either shrunk to a point or deformed into each other without intersecting the impenetrable torus (and the flux ϕ). If multi-valued wavefunctions are used one has $\Psi(2,1) = \Psi(1,2)\exp(\mp i\gamma)$ for the upper and middle parts of figure 3, respectively, while $\Psi(2,1) = \Psi(1,2)\exp(-2i\gamma)$ for the lower part (as before $\gamma = e\phi/\hbar c$ where ϕ is the magnetic flux inside the toroidal solenoid). This situation strongly resembles that of the cylindrical solenoid, but there are more topologically non-equivalent possibilities. As for the cylindrical solenoid the double exchange of particles does not in general lead to the initial wavefunction. For example, under the double exchange composed of single exchanges (each in the clockwise direction), presented in the upper part of figure 3, the wavefunction acquires the factor $\exp(-2i\gamma)$.

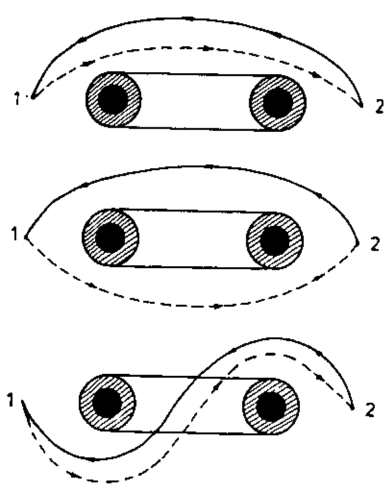

Figure 2. The trivial exchange paths in the field of a toroidal solenoid.

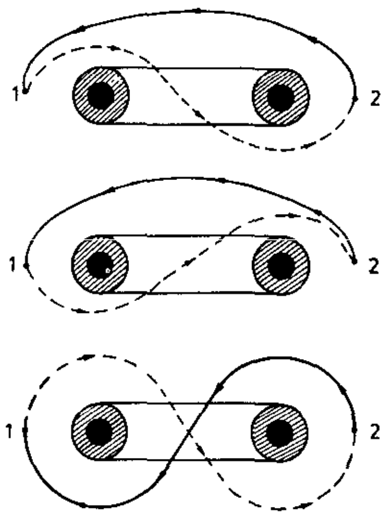

Figure 3. The non-trivial exchange paths in the field of a toroidal solenoid.

6. Interacting torons

Now we return to interacting torons. Consider first the simplified case when the symmetry axes of toroidal solenoids 1 and 2 are parallel to the z-axis. In addition, the solenoids are assumed to be thin ($R_1 \ll d_1$, $R_2 \ll d_2$). Then, it follows from (3.1) that outside the solenoids one has

$$
\begin{aligned}
A_{21z} &= \frac{1}{2\pi} \frac{\phi_1 d_1}{(d_1\rho)^{3/2}} \frac{1}{\sinh \nu_1} (\rho Q^1_{1/2}(1) - d_1 Q^1_{-1/2}(1)) \\
A_{21x} &= \frac{x}{\rho} A_{21\rho} \qquad A_{21y} = \frac{y}{\rho} A_{21\rho} \\
A_{21\rho} &= -\frac{1}{2\pi} \phi_1 d_1 \frac{z}{(d_1\rho)^{3/2}} \frac{1}{\sinh \nu_1} Q^1_{1/2}(1).
\end{aligned}
\tag{6.1}
$$

Here $Q^\sigma_1(1) \equiv Q^\sigma_1(\cosh \nu_1)$, $\sinh \nu_1 = (r^2 + d_1^2)/2d_1\rho$, $x = x_1 - x_2$, etc. $\rho = \sqrt{x^2+y^2}$, $r = (x^2+y^2+z^2)^{1/2}$. The vector potential $\boldsymbol{A}_{12}$ is obtained from $\boldsymbol{A}_{21}$ by the interchanging of particle subscripts 1 and 2. The vector potential thus obtained are symmetrical with respect to interchange of particle coordinates:

$$
\boldsymbol{A}_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \boldsymbol{A}_{12}(\boldsymbol{r}_2, \boldsymbol{r}_1) \qquad A_{21}(\boldsymbol{r}_1, \boldsymbol{r}_2) = A_{21}(\boldsymbol{r}_2, \boldsymbol{r}_1). \tag{6.2}
$$

The vector potentials $\boldsymbol{A}_{12}$ and $\boldsymbol{A}_{21}$ may be expressed as gradients of multi-valued functions $\chi_{12}(\boldsymbol{r})$ and $\chi_{21}(\boldsymbol{r})$:

$$
\boldsymbol{A}_{21} = \mathrm{grad}_{\boldsymbol{r}}\, \chi_{21} \qquad \boldsymbol{A}_{12} = \mathrm{grad}_{\boldsymbol{r}}\, \chi_{12}.
$$

The functions χ_{21} and χ_{12} are obtained from (3.3) by making the substitution $\phi \to \phi_1$, $\mu_0 \to \mu_1$, $ga \to g_1 a_1 = (\phi_1/2\pi)(\coth \mu_1 - 1)^{-1}$ for χ_{21} and $\phi \to \phi_2$, $\mu_0 \to \mu_2$, $ga \to g_2 a_2 = (\phi_2/2\pi)(\coth \mu_2 - 1)^{-1}$ for χ_{12}. The linear combination $e\boldsymbol{A} \equiv e_1 \boldsymbol{A}_{12}(\boldsymbol{r}) - e_2 \boldsymbol{A}_{21}(\boldsymbol{r})$ entering into (2.3) may be presented in the form

$$
e\boldsymbol{A} = \mathrm{grad}_{\boldsymbol{r}}\, e\chi \qquad e\chi \equiv e_1 \chi_{12}(\boldsymbol{r}) - e_2 \chi_{21}(\boldsymbol{r}).
$$

Now we can write the classical Lagrangian (2.2) in the gauge invariant form

$$L = \tfrac{1}{2}m_1 v_1^2 + \tfrac{1}{2}m_2 v_2^2 + \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\left(e_1\chi_{12}(\boldsymbol{r}) - e_2\chi_{21}(\boldsymbol{r})\right) - V.$$

The unitary transformation

$$\Psi = \Psi' \exp\left(\frac{\mathrm{i}e\chi}{\hbar c}\right)$$

eliminates the vector potential from (2.3). Consider particular cases. Let the toron parameters be the same ($m_1 = m_2 = m$, $e_1 = e_2 = e$, $d_1 = d_2 = d$, $R_1 = R_2 = R$) except for the magnetic fluxes ϕ_1 and ϕ_2. Then,

$$\chi_{12} = \phi_2\chi(\boldsymbol{r}) \qquad \chi_{21} = \phi_1\chi(\boldsymbol{r})$$

where χ is obtained from (3.3) by dropping the overall factor ϕ. Thus

$$e_1\chi_{12} - e_2\chi_{21} = e(\phi_2 - \phi_1)\chi$$
$$\Psi = \Psi' \exp\left(\frac{\mathrm{i}e(\phi_2 - \phi_1)}{\hbar c}\chi\right). \tag{6.3}$$

In addition, let $\phi_1 = \phi_2$. Then $\boldsymbol{A}_{21} = \boldsymbol{A}_{12}$ and $\Psi = \Psi'$. From (2.4) or (2.5) it follows that the vector potential drops out from the SE. This means that the presence of vector potentials outside the torons with $\phi_1 = \phi_2$ changes neither the dynamics of torons nor their exchange properties. If $\phi_1 = -\phi_2 = \phi$ then (6.3) gives

$$\Psi = \Psi' \exp(-2\mathrm{i}\gamma\chi) \qquad \gamma = e\phi/\hbar c. \tag{6.4}$$

If Ψ is chosen to be single-valued, then Ψ' suffers the discontinuity

$$\Psi'(\rho \leqslant d - R, z = 0-) = \exp(2\mathrm{i}\gamma)\Psi'(\rho \leqslant d - R, z = 0+).$$

It should be noted that the equality $\phi_1 = -\phi_2$ does not mean that anyons are different. To see this we turn to figure 4. At the upper part of this figure we see two identical

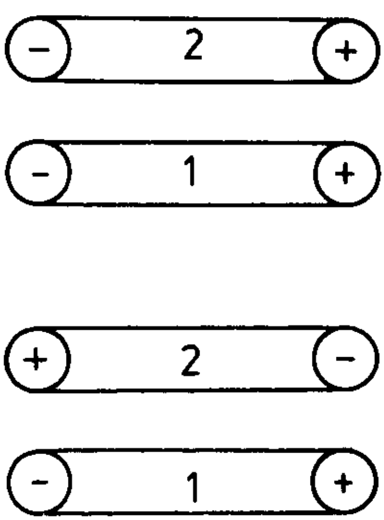

Figure 4. Two toroidal solenoids with equal (upper part) and opposite (lower part) magnetic fluxes.

toroidal solenoids with $\phi_1 = \phi_2$. The signs + and − mean that the magnetic field ($\boldsymbol{H} = \boldsymbol{e}_\varphi g/\rho$, $g = \phi[2\pi(d - \sqrt{d^2 - R^2})]^{-1}$) is directed from and towards the observer, respectively. It has the same direction in both the solenoids. Now we begin to rotate the second solenoid around the axis normal to the plane of the figure. After rotation at the angle π is performed, we obtain the situation shown at the lower part of the same figure. We observe that the magnetic field in the second solenoid has been changed to lie in the opposite direction. This means that for an external observer the magnetic flux of the second toron has changed its sign and that torons with $\phi_1 = \phi_2$ are indeed the same. Now we try to exchange torons with $\phi_1 = -\phi_2$. Firstly, however, we must know how the relative toroidal coordinates μ, θ and φ entering into χ_{12} and χ_{21} behave under the particle exchange. It follows from (3.2) that to $\boldsymbol{r}_1 \leftrightarrow \boldsymbol{r}_2$ there corresponds $\mu \to \mu$, $\theta \to -\theta + 2\pi n$, $\varphi \to \varphi + (2m+1)\pi$. This leads to the following change of χ:

$$\chi(-\boldsymbol{r}) = -\chi(\boldsymbol{r}) - n.$$

If the wave function Ψ is chosen to be symmetrical, then this results in the following behaviour of Ψ' under the particle exchange:

$$\Psi'(2,1) = \Psi'(1,2)\exp(-4\mathrm{i}\gamma\chi(\boldsymbol{r}) - 2\mathrm{i}\gamma n). \tag{6.5}$$

We see that exchange properties of torons depend essentially on their relative positions and orientations. For particular cases we obtain a situation similar to that of cyons. Let torons 1 and 2 be in such a relative position that $\chi(1,2) = 0$. From the explicit expression for χ (see (3.3)) it follows that this occurs, for example, for $\theta = 0$. This in turn happens either when the equatorial planes of the solenoids lie in the same plane (this corresponds to $z = z_1 - z_2 = 0$ in (3.2)), or when the torons are separated enough. The latter is due to the fact that toroidal angle θ decreases at large distances ($\theta \approx (2\sqrt{d^2 - R^2}/r)\cos\theta_s$ for $r \to \infty$). In this case (6.5) reduces to

$$\Psi'(2,1) = \Psi'(1,2)\exp(-2\mathrm{i}\gamma n).$$

From this we obtain Bose, Fermi or intermediate statistics depending on the value of γ ($\equiv e\phi/\hbar c$).

The fact that statistical properties of anyons can depend on their mutual separation is not new. For two-dimensional anyons this has recently been proved in a very interesting paper, [18] (in the framework of the Chern–Simons gauge theory). When the torons' dimensions tend to zero we obtain the magnetic toroidal moment [19] with electric charge attached. The unusual statistics is obtained for anyons with opposite toroidal moments. There is an intuitive explanation of such a different wavefunction behaviour for $\phi_1 = \phi_2$ and $\phi_1 = -\phi_2$. When torons with $\phi_1 = \phi_2$ pass through each other (the upper part of figure 4) the net change of the wavefunction phase equals zero. In fact, the charge particle of toron 2 passing through the hole of toron 1 contributes the value γ to the phase while the particle of 1 passing through the hole of 2 contributes $-\gamma$. When $\phi_1 = -\phi_2$ (the lower part of the same figure) both particles contribute the same phase γ.

When the symmetry axes of the torons have arbitrary orientation (i.e. they are neither parallel nor antiparallel) one should use in (2.3) vector potentials defined as

$$\begin{aligned} \tilde{A}_{21i} &= \sum R_{iK}(\varphi_1, \theta_1, \psi_1) A_{21K} \\ \tilde{A}_{12i} &= \sum R_{iK}(\varphi_2, \theta_2, \psi_2) A_{12K}. \end{aligned} \tag{6.6}$$

The angles φ, θ, ψ define the orientation of the particular toron with respect to a fixed laboratory frame. The vector potentials $\boldsymbol{A}_{21}$ and $\boldsymbol{A}_{12}$ in the RHS of (6.6) are defined by (6.1).

7. Ambiguities arising from various choices of gauge

At first we demonstrate arising uncertainties using two interacting cyons as an example. The vector potential of the cylindrical solenoid in a Coulomb gauge is equal to $\boldsymbol{A}=\boldsymbol{e}_\varphi\phi\rho/2\pi R^2$ inside the solenoid ($\rho<R$) and $\boldsymbol{A}=\boldsymbol{e}_\varphi\phi/2\pi\rho$ outside it ($\rho>R$). It falls as ρ^{-1} at large distances from the axis. The magnetic field equals $\boldsymbol{H}=\boldsymbol{e}_z\phi/\pi R^2$ inside the solenoid and zero outside it. On the other hand, one may equally use the following vector potentials [10, 11, 20]: $A'_x=0$ everywhere, $A'_y=\phi(x+\sqrt{R^2-y^2})/\pi R^2$ inside the solenoid. Outside, it differs from zero within the hatched strip ($-R<y<R, x>\sqrt{R^2-y^2}$) shown in figure 5. There, it equals $2\phi\sqrt{R^2-y^2}/\pi R^2$. This vector potential generates the same magnetic field as A. For the infinitely thin solenoid it reduces to $A'_x=0$, $A'_y=\phi\theta(x)\delta(y)$. Now we return to interacting cyons. Inserting $\boldsymbol{A}'$ into (2.3) we obtain the following net cyon vector potentials entering there:

$$A_x=0 \qquad \text{everywhere}$$

$$A_y=\begin{cases}\dfrac{2\phi x}{\pi R^2} & \text{for } \rho<R\\[2ex] \dfrac{2\phi}{\pi R^2}\sqrt{R^2-y^2}\,[\theta(x-\sqrt{R^2-y^2})-\theta(-x-\sqrt{R^2-y^2})] & \text{for } \rho>R.\end{cases}$$

(Here x, y and ρ are relative coordinates.) This means that inside the total strip composed as shown in figure 5, and symmetrical to it, $A_y=\pm 2\phi\sqrt{R^2-y^2}/\pi R^2$ for right and left half-strips, respectively. For the infinitely thin solenoid $A_y=\phi\delta(y)\theta(x)-\theta(-x))$. From this it follows that anyons interact only if their relative coordinates lie in the hatched strip. Such a distinct behaviour of $\boldsymbol{A}$ and $\boldsymbol{A}'$ means that one should not pay too much attention to the particular realization of vector potentials. According to Wu and Yang [21] only the phase factor $\exp((\mathrm{i}e/\hbar c)\oint A_\mu\,\mathrm{d}x_\mu)$ is physically meaningful and measurable. In fact, it is the same for $\boldsymbol{A}$ and $\boldsymbol{A}'$. The vector potentials $\boldsymbol{A}$ and $\boldsymbol{A}'$ are connected by the gauge transformation. The corresponding wavefunctions Ψ and Ψ' are connected by the unitary transformation. This means that all observables are the same for Ψ and Ψ'.

Going over to interacting torons we observe that in addition to the vector potential $\boldsymbol{A}$ of the toroidal solenoid discussed in section 3 there exists $\boldsymbol{A}'$ [10, 11], the single non-vanishing component of which (A'_z) differs from zero in the vincinity nearest to the toroidal solenoid. It equals $g\ln(d+\sqrt{R^2-z^2})/\rho$ inside the solenoid and $g\ln(d+\sqrt{R^2-z^2})/(d-\sqrt{r^2-z^2})$ outside it, in the hatched region (see figure 6). It is zero in

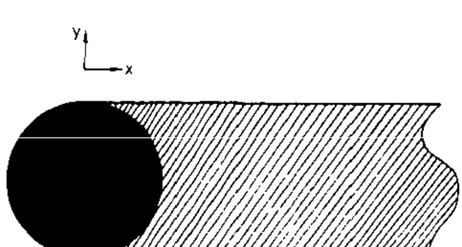

Figure 5. The vector potential of cylindrical solenoids in a non-standard gauge. Outside the solenoid it differs from zero in the hatched region only.

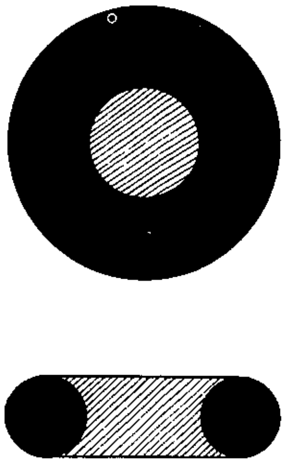

Figure 6. The same as in figure 5 but for a toroidal solenoid.

other space regions. For the infinitely thin solenoid ($R \ll d$) it reduces to $A'_z = \phi\delta(z)\theta(d-\rho)$ [22]. The total net vector potential for two identical interacting torons ($\boldsymbol{A}' = \boldsymbol{A}'_{12}(\boldsymbol{r}) - \boldsymbol{A}'_{21}(-\boldsymbol{r})$, $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$) equals zero if $\phi_1 = \phi_2$ and twice the value of A'_z if $\phi_1 = -\phi_2$. This means that in such a gauge the torons interact only if their relative coordinates lie in the hatched region. It is easy to check that both $\boldsymbol{A}$ and $\boldsymbol{A}'$ satisfy the condition that $\oint A_l \, \mathrm{d}xl = \phi$ for closed contours passing through the solenoid's hole and zero otherwise. The vector potentials $\boldsymbol{A}$ and $\boldsymbol{A}'$ are connected by the gauge transformation $\boldsymbol{A} = \boldsymbol{A}' + \operatorname{grad} \partial\alpha/\partial z$. For the infinitely thin ($R \ll d$) torons the function α equals [11, 23] $(1/2\pi)\phi \int\int (\mathrm{d}x_1 \, \mathrm{d}y_1/|\boldsymbol{r}-\boldsymbol{r}_1|)$. Here, integration is performed over the circle of the radius d lying in the $z=0$ plane. The double integral may be expressed through the linear one [23]: $\int\int |\boldsymbol{r}-\boldsymbol{r}_1|^{-1} \, \mathrm{d}x_1 \, \mathrm{d}y_1 = 2\pi(\sqrt{z^2+d^2} - |z|) - 2\sqrt{d} \int_0^\rho \mathrm{d}x \, x^{-1/2} \times Q_{1/2}[(r^2+x^2+d^2)/2 \, \mathrm{d}x]$. The corresponding wavefunctions are connected by the singular (for the infinitely thin torons) but unitary transformation $\Psi = \Psi' \exp[(\mathrm{i}e/\hbar c)(\partial\alpha/\partial z)]$. Hence it follows that Ψ and Ψ' behave differently under particle exchange (in spite of the fact that they correspond to vector potentials with the same circulation). The main conclusion of this section is that exchange properties of the wavefunctions depend on the particular gauge choice. Thus, some caution is needed in their interpretation.

8. Discussion

Here we analyse the frequently used assertion [15–17] that there are no non-trivial exchange paths in three-dimensional space. Usually, one starts with the consideration of a plane with a singular isolated point P in it. It turns out that for a charged particle multi-valued wavefunctions are allowed if this point carries magnetic flux ϕ. In fact, a closed contour embracing P cannot be shrunk to a point without intersecting P. Going over to three-dimensional space, one encounters the following alternative. First, one may continue to treat P as an isolated singular point. In this case the above contour may be shrunk to a point without intersecting P (for this, one at first rotates half of the contour around the axis lying in the initial plane and passing through P). Therefore, multi-valued wavefunctions are not allowed. On the other hand, one may treat P as a trace of an infinite singular line $\mathscr{L}$ piercing the plane at P. The contour encircling

$\mathscr{L}$ cannot be contracted without intersecting it. Multi-valued wavefunctions are allowed if $\mathscr{L}$ carries the magnetic flux ϕ, thus coinciding with an infinitely thin cylindrical solenoid. Let us have on the plane two singular points with $\phi_1 = -\phi_2$. This can be viewed as the traces of two parallel singular lines which pierce the plane at those points. For the charged particle one easily recovers topologically trivial (which embrace either both or none of the solenoids) and non-trivial (which embrace one of the solenoids) exchange paths [24]. Physically, these singular lines can be realized as two cylindrical solenoids with $\phi_1 = -\phi_2$. The charge particle scattering on them was studied in [11, 13, 25]. The singular line may also have a form of the circular filament which carries the magnetic flux ϕ and which may be viewed as an infinitely thin toroidal solenoid. For the charged particle, multi-valued wavefunctions are allowed as the closed contours (passing the solenoid's hole) exist which cannot be shrunk to a point. The above singular lines may be considered as the limiting cases of the finite impenetrable cylindrical and toroidal solenoids shown in figures 1–3. So far we have considered the behaviour of charge particles in the field of cylindrical and toroidal solenoids. Now we turn again to toroidal anyons. We have seen in section 6 that they exhibit fractional statistics with respect to their exchange. This contradicts the frequently occurring assertion (e.g. see, review [26] and references therein) that exotic statistics do not exist if the number of spatial dimensions is greater than two. The proof grounds essentially on the fact that after the removal of the points corresponding to the coinciding particle coordinates the remaining portion of space is multiconnected for $d = 2$ and simply connected for $d \geqslant 3$ [27, 28]. In the treated case the part of space occupied by the coinciding identical torons is isomorphic to the torus. The remaining portion of space (lying outside that torus) is the multiconnected one. The dimensions of the toron may be arbitraily small, yet there remains a finite possibility for one particular toron to penetrate through the hole of the other. This is the reason for the appearance of non-standard statistics in the three-dimensional case. The reasoning of the cited references fails as the particles there were thought of as point-like structureless objects. The non-standard statistics disappears when the hole of a toron is closed. The smallness of the toron is not essential as the mutual penetration of torons does not depend on their dimensions. Only their non-trivial topology is important. The question arises of how to choose the mutual orientation of the interacting torons. The angles describing this orientation enter into the hamiltonian as parameters. The reason for this is that the kinetic energy of the particles is taken to coincide with that for point particles. For the toron (however small its dimensions) the kinetic energy should depend on the orientation angles and corresponding momenta (like for the quantum symmetrical top). At the present stage of investigation the mutual orientation angles may be chosen from minimum energy considerations. We do not intend to elaborate further these points here.

It would be appropriate to mention that there are three-dimensional objects exhibiting fractional statistics: the so-called dyons, which are composites of a monopole and a charged particle [6, 8]. Finally, there exists an excellent three-dimensional description [29] of the quantum Hall effect which does not appeal to its two-dimensional nature.

9. Conclusion

The author, being a non-specialist in solid state physics, cannot appreciate the practical meaning of the results obtained. Probably, they have some relation to the recently observed fractional quantum Hall effect in three-dimensional structures [30].

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